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Scalar field theory on fuzzy S^4

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Abstract: Scalar fields are studied on fuzzy S^4 and a solution is found for the elimination of the unwanted degrees of freedom that occur in the model. The resulting theory can be interpreted as a Kaluza-Klein reduction of $\mathbb{CP}^3$ to S^4 in the fuzzy context.

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Contents

1. Introduction

To access the physics of field theories in the strong coupling regime non-perturbative methods are necessary. These typically involve the reduction of the field theory to a model with a finite number of degrees of freedom and the use of numerical methods to sample the possible configurations. The now standard method in this regard is lattice field theory. It has been developed in the last twenty years into a refined tool [1].

However, resorting to a lattice is not the only possible method of reducing a field theory to a finite number of degrees of freedom. An alternative is what has become known as the fuzzy approach [2]–[12], see [4] for a review.

In this approach one takes the underlying space in the form of its algebra of functions and seeks a sequence of non-commutative algebras with finite dimensional representations, whose limiting form reproduces the commutative algebra. The elements of the algebra can then be represented by matrices and will play the role of scalar fields in an approximation to field theory. Dirac and Yang-Mills fields on fuzzy spaces have also been considered. These involve projective modules over the algebra [3]. The method has further been extended to superspace and a fuzzy supersphere has been constructed [13].

As a non-perturbative approach to field theory the fuzzy scheme is very different from the lattice one and many new features emerge. The most surprising is perhaps the phenomenon of UV/IR mixing [5, 6, 7] where a residue of the microscopic non-commutativity remains in the commutative limit. This can be eliminated by suitably modifying the original action of the model.

The most intensely studied example in the fuzzy approach is the so called fuzzy sphere, S_F^2, which is realized as the matrix algebra of dimension $L+1$, denoted here Mat_{L+1}, with the inner product $(M,N)=\frac{\mathrm{Tr}}{L+1}(M^\dagger N)$, where M and $N\in\mathrm{Mat}_{N+1}$ and the geometry is specified through the set of derivations $\mathcal{L}_i = AdL_i$ that correspond to the adjoint action of the generators L_i of SU(2) (with spin $s = L/2$). For scalar fields only the laplacian (or sequence of laplacians parametrized by the matrix size) plays a role and the geometry is specified by a choice of laplacian. Choosing the laplacian to be $\mathcal{L}^2$ gives us a round sphere. One can therefore view the round fuzzy sphere, S_F^2, in the spirit of Connes[1] [14] as the triple $(\mathrm{Mat}_{L+1}, \mathcal{L}^2, (\cdot,\cdot))$ i.e. given by the algebra Mat_{L+1} together with the differential operator $\mathcal{L}^2$ and the inner product $(M, N) = \frac{\mathrm{Tr}}{L+1}(M^\dagger N)$. More generally one could consider a non-round sphere by utilizing an alternative expression for the laplacian.

For S_F^2 the algebra Mat_{L+1} can be viewed as generated by $X_i = \frac{R}{\sqrt{s(s+1)}} L_i$, which satisfy

$$\sum_{i=1}^{3} X_i X_i = R^2 \mathbf{1}\,. \tag{1.1}$$

The X_i satisfy the commutation relations

$$[X_i, X_j] = \frac{iR}{\sqrt{s(s+1)}} \epsilon_{ijk} X_k \tag{1.2}$$

and become commutative in the large L, fixed R, limit so that we recover the commutative algebra of functions on S^2. If instead one scales R with L one can obtain the non-commutative plane [7, 15].

The construction above can be carried out for any $\mathbb{CP}^N$ [8] and will be reviewed from the above point of view in the next section. One can imagine carrying the prescription out more generally by specifying the family of matrices and laplacians such that one recovers the approximated space up to isospectral equivalence. However, examples beyond the most symmetric spaces have not yet been constructed in this way.

The known constructions (to date) of matrix approximations to continuum spaces are almost exclusively coadjoint orbits and their products. The one apparent exception[2] is S^4 (see [9]–[12]). This is an especially important example since it is the most natural replacement of $\mathbb{R}^4$ in studies of euclidean quantum field theory. It is also unusual in that S^4 does not admit a symplectic structure and hence a fuzzy version of it could not be achieved by quantization of the classical space. It is, therefore, important to clarify in what sense a matrix approximation to S^4 exists.

We will demonstrate that the matrix algebra approximation of S^4 under discussion is really a matrix version of $\mathbb{CP}^3$ in disguise. The desired S^4 emerges from a Kaluza-Klein type construction in the fuzzy context.

Curiously, the space $\mathbb{CP}^3$ has recently been considered [18] in the context of higher dimensional quantum Hall liquids and perhaps some of the techniques developed here may be of use in the present setting.

The structure of the paper is as follows: In section 2 we present an overview of the fuzzy approach in the case of $\mathbb{CP}^N$. In section 3 we show that S^4 emerges from a matrix construction as the matrix size is sent to infinity. In section 4 we study the representation

[1]Note: Connes, in contrast the construction here, uses a Dirac Operator.

[2]See also the related construction of fuzzy even [16] and odd [17] spheres.

JHEP11(2003)051

content of the matrix algebra and give our solution to the problem of suppressing non-S^4 modes. As a side product of our construction we are able to find the projector necessary to eliminate non-S^4 modes. Section 5 gives our prescription for the scalar field action where the non-S^4 modes are dynamically suppressed. One can see from the general discussion that it corresponds to a $\mathbb{C}\mathrm{P}^3$ model where the maximal SO(6) symmetry is reduced to an SO(5) symmetry. Geometrically the model corresponds to a Kaluza-Klein type space which is an S^2 bundle over S^4 with the radii of the S^2 fibres being sent to zero as the energy scale of the unwanted modes is sent to infinity. Section 6 gives our conclusions.

2. Construction of $\mathbb{C}\mathrm{P}^N_F$

We begin by reviewing the construction of $\mathbb{C}\mathrm{P}^N_F$. For this one takes the L fold symmetric tensor product of the fundamental representation of $\mathrm{SU}(N+1)$, which in terms of Young tableaux is denoted

$$\mathbf{d_L^N} = \underbrace{\square\ldots\square}_{L} \tag{2.1}$$

with the dimension $d_L^N = \frac{(L+N)!}{N!L!}$. The sequence of matrix algebras under consideration will then be $\mathrm{Mat}_{d_L^N}$ which will be endowed with the inner product $(M,N) = \frac{\mathrm{Tr}}{d_L^N}(M^\dagger N)$ with M and $N \in \mathrm{Mat}_{d_L^N}$.

The geometry has not as yet been specified since a sequence of differential operators is not yet given. When the geometry is specified by a laplacian, the number of eigenvalues of that laplacian, less than a cutoff momentum scale Λ, increases with the cutoff as Λ^d. Since the total number of degrees of freedom in the matrix algebra is $(d_L^N)^2$ and our cutoff is L, we can deduce the dimension of the space being approximated via

$$d = 2\lim_{L\to\infty}\frac{\ln d_L^N}{\ln L} = 2N \tag{2.2}$$

which is consistent with $\mathbb{C}\mathrm{P}^N$.

If we choose our differential operators to be built from the generators, T_a, of $\mathrm{SU}(N+1)$ in the $\mathbf{d}_L^N$ representation and the generators $-T_a^R$ in the complex conjugate representation, with these latter operators acting on matrices on the right, then matrices are equivalent to the product representation

$$\underbrace{\square\ldots\square}_{L}\otimes\underbrace{\overline{\square\ldots\square}}_{L} \tag{2.3}$$

which can be expanded in terms of $\mathrm{SU}(N+1)$ representations to yield the expansion of matrices in terms of polarization tensors. For example for SU(4) with $L=3$ we have

$$\overline{\ydiagram{3}} = \ydiagram{3,3,3}$$

and the expansion:

$$\ydiagram{3}\otimes\ydiagram{3,3,3} = \mathbf{1}\oplus\ydiagram{2,1,1}\oplus\ydiagram{4,2,2}\oplus\ydiagram{6,3,3}. \tag{2.4}$$

In the special case of SU(2), for example with $L=3$, we have

$$\overline{\ydiagram{3}} = \ydiagram{3}$$

and the decomposition

$$\square\square\square \otimes \square\square\square = \mathbf{1} \oplus \square\square \oplus \square\square\square\square \oplus \square\square\square\square\square\square \tag{2.5}$$

and we see that the expansion is in terms of integer angular momentum and is cut off at $l = 3$. In general, for the L-fold symmetric tensor product representation, the angular momentum will be cutoff at $l = L$ and as $L \to \infty$ we recover all of the representations corresponding to functions on S^2. Similarly in the $L \to \infty$ limit of the $\mathrm{SU}(N+1)$ case we recover $\mathbb{C}\mathrm{P}^N$.

Fuzzy $\mathbb{C}\mathrm{P}^N$ would then, by analogy with S^2_F, be considered as the triple $(\mathrm{Mat}_{d^N_L}, \mathcal{L}^2, (\cdot,\cdot))$ where now the sequence of matrices is that of dimension $d^N_L = \frac{(L+N)!}{N!L!}$, $\mathcal{L}^2 = (AdT_a)^2$ with as before $(M, N) = \frac{\mathrm{Tr}}{d^N_L}(M^\dagger N)$, where M and $N \in \mathrm{Mat}_{d^N_L}$.

A scalar field action on $\mathbb{C}\mathrm{P}^N$ would then correspond to

$$S[\Phi] = \frac{\mathrm{Tr}}{d^N_L}\left[\frac{1}{2}\Phi\mathcal{L}^2\Phi + V(\Phi)\right] \tag{2.6}$$

The ϕ^4 model of this type on S^2_F has been analyzed in [6] and is currently being studied numerically as a testing ground for the feasibility of the numerical approach.

3. Construction of S^4_F

Let us now turn to the construction of S^4_F. For this observe that $X_a = \frac{R}{\sqrt{5}}\Gamma_a$, with Γ_a the Dirac matrices including Γ_5, satisfy

$$\sum_{a=1}^{5} X_a X_a = R^2\mathbf{1}\,. \tag{3.1}$$

This gives us a matrix approximation to the defining equation of S^4 in $\mathbb{R}^5$ and is the direct analogue of (1.1) for S^2_F. To get our sequence of matrices approximating S^4 we are led to consider representations of the group Spin(5) (or equivalently Sp(2)). In the above we have used the defining four dimensional representation $(\frac{1}{2}, \frac{1}{2})$ of Spin(5). We can therefore consider the irreducible representation obtained from the L fold symmetric tensor product of this representation i.e. the Spin(5) representation $(\frac{L}{2}, \frac{L}{2})$ which will contain a set of five matrices: J_a, $a = 1, \ldots, 5$ which can be realized as the symmetrization of L copies of the Γ matrices in the Spin(5) fundamental representation:

$$J_a = \Big(\underbrace{\Gamma_a\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{\text{L-times}} + \mathbf{1}\otimes\Gamma_a\otimes\cdots\otimes\mathbf{1} + \cdots + \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\Gamma_a\Big)_{\mathrm{sym}} \tag{3.2}$$

where the subscript sym indicates that we are projecting onto the irreducible totally symmetrized representation. These matrices satisfy the relation

$$J_a J_a = L(L+4)\mathbf{1}$$

JHEP11(2003)051

so that we can define a sequence of matrices, X_a, given by

$$X_a = \frac{R}{\sqrt{L(L+4)}} J_a \, . \tag{3.3}$$

The algebra generated by these functions will become commutative in the infinite L limit (see the discussion associated with eq. (4.5)) and since in the commutative case it is easy to check that the five co-ordinate functions x_a (satisfying $x_a^2 = R^2$) generate $C^\infty(S^4)$ we will recover S^4 in the limit.

If we return to the matrices arising at level $L = 1$, we see that the Γ_a are not sufficient to form a basis for all matrices, but rather, to get a basis, we need to include the matrices $\sigma_{ab} = \frac{1}{2i}[\Gamma_a, \Gamma_b]$. We can now expand any 4×4 matrix in terms of the 16 matrices $\{\mathbf{1}, \Gamma_a, \sigma_{ab}\}$. In this approximation (i.e. $L = 1$) a matrix representing a function on S^4 will be of the form

$$F = F_0 \mathbf{1} + F_a \Gamma_a \tag{3.4}$$

and our cutoff angular momentum on S^4 will again be $l = 1$ (as it was at the corresponding $L = 1$ approximation to S^2). However, matrices with vanishing coefficients of σ_{ab} (as in (3.4)) do not close to form a subalgebra, but if we include such coefficients the corresponding matrices will have no counterpart in the expasion of functions on S^4. One option as argued by Ramgoolam [11] is to project out such terms, in which case one is left with a non-associative algebra. This involves additional complications and does not seem particularly suited to numerical work. In addition the necessary projector must be constructed. We will return to this point in a concluding section where we will, in fact, give the projector.

An alternative is to include arbitrary coefficients of σ_{ab} (demanding an associative algebra) and attempt to suppress such coefficients of unwanted terms, by making their excitation improbable in the dynamics. In this approach our algebra will be a full matrix algebra and obviously associative. The principal task of this paper will therefore be to give a prescription for suppressing the additional modes that arise in this extended algebra.

From a physics point of view the matrices are to play the role of our scalar fields which will be sampled in a Monte-Carlo simulation. We will therefore be seeking an appropriate scalar field action which suppresses the non-S^4 modes in a probabilistic sense in our simulations.

A successful method of suppressing the unwanted modes would be to add to the scalar action a term $S_I[\Phi]$ which is positive for any Φ and zero only for matrices that correspond to functions on S^4, and non-zero for those that do not. The modified action would therefore be of the form $S[\Phi] + hS_I[\Phi]$. The parameter h should then be chosen large and positive. The probability of any given matrix configuration then takes the form

$$\mathcal{P}[\Phi] = \frac{\mathrm{e}^{-S[\Phi] - hS_I[\Phi]}}{Z} \tag{3.5}$$

where

$$Z = \int d[\Phi] \mathrm{e}^{-S[\Phi] - hS_I[\Phi]} \tag{3.6}$$

is the partition function of the model.

We will show that this can be achieved by choosing $S_I[\phi] = \frac{\mathrm{Tr}}{d_L}(\frac{1}{2}\Phi\Delta_I\Phi)$ with $d_L = \frac{(L+1)(L+2)(L+3)}{6}$ and Δ_I a positive operator. It can be interpreted as a modification of the laplacian which is zero on matrices corresponding to S^4.

From (2.2) we see that the dimension of the space being approximated by this sequence of matrices is in fact six and not four. We will further see that the entire model, when the unwanted degrees of freedom are included, corresponds to a fuzzy version of $\mathbb{C}\mathrm{P}^3$. $\mathbb{C}\mathrm{P}^3$ is an S^2 bundle over S^4 and the parameter h can be related to the radius of the S^2 fibres over S^4 with the radius being sent to zero as $h \to \infty$ [19]. In this sense we will have a fuzzy Kaluza-Klein type space whose low energy limit is S^4.

4. The representation content and laplacian

The matrices $\sigma_{ab}/2$ with $a, b = 1, \ldots, 5$ are the generators of Spin(5) in the fundamental representation. If we further identify $\Gamma_a = \sigma_{a6} = -\sigma_{6a}$, then the set $\sigma_{AB}/2$ with $A, B = 1, .., 6$ are the generators of Spin(6) in its fundamental representation $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. We can similarly identify J_{AB} in the L fold symmetric tensor product representation, by replacing Γ_a in (3.2) by $\sigma_{AB}/2$. The resulting set J_{AB} satisfy the algebra:

$$[J_{AB}, J_{CD}] = \imath\,(\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{AD}J_{BC} - \delta_{BC}J_{AD}) \tag{4.1}$$

and generate the Spin(6) irreducible representation $(\frac{L}{2}, \frac{L}{2}, \frac{L}{2})$ with dimension $d_L = \frac{(L+1)(L+2)(L+3)}{6}$. The subset J_{ab}, are Spin(5) generators in the $(\frac{L}{2}, \frac{L}{2})$ representation and the subset $J_{a6} = J_a/2$ transform as a vector under Spin(5) in this representation.

We can therefore view the 4×4 matrix algebra as the tensor product $(\frac{1}{2}, \frac{1}{2}) \otimes \overline{(\frac{1}{2}, \frac{1}{2})}$ if we take a Spin(5) point of view or as $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \otimes \overline{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}$ from a Spin(6) perspective. Similarly, for the L dependent sequence of matrices[3] one can take either a Spin(5) or Spin(6) view of the matrix algebra. The dimension of both sequences of representations is $d_L = \frac{(L+1)(L+2)(L+3)}{6}$ and the sequence of matrix algebras under consideration is Mat_{d_L}. From (2.2) we see that this sequence is one associated with an approximation to a six rather than four dimensional space. Since Spin(6) $\cong$ SU(4) the natural geometry associated with the Spin(6) approach is that of a fuzzy approximation to $\mathbb{C}\mathrm{P}^3$.

In fact all the representations under consideration here can also be considered as representations of Spin(6) $\cong$ SU(4). Note, the L fold symmetric tensor product representation $(\frac{L}{2}, \frac{L}{2}, \frac{L}{2})$ of Spin(6) is precisely the representation

$$\underbrace{\square \ldots \square}_{L}$$

or equivalently the $(L, 0, 0)$ representation of SU(4).

In this sequence of representations the Spin(5) generators J_{ab} are still of the form

$$J_{ab} = \frac{1}{4i}[J_a, J_b] \quad \text{a,b=1,...,5} \tag{4.3}$$

[3] Note: for Spin(5) we have $\overline{(\frac{L}{2}, \frac{L}{2})} = (\frac{L}{2}, \frac{L}{2})$ while for Spin(6) we have $\overline{(\frac{L}{2}, \frac{L}{2}, \frac{L}{2})} = (\frac{L}{2}, \frac{L}{2}, -\frac{L}{2})$.

as can be seen from (4.1). It is easy to verify that

$$J_{ab}J_{ab} = L(L+4)\mathbf{1}\,. \tag{4.4}$$

So the commutator of the coordinate matrices defined in (3.3) is given by

$$[X_a, X_b] = 4iR^2 \frac{J_{ab}}{L(L+4)}\,. \tag{4.5}$$

In the $L \to \infty$ limit with R fixed the right hand side of (4.5) goes to zero, and the coordinates commutes. Therefore, since we have retained the constraint (3.1), we recover *commutative* S^4.

We have not as yet defined a geometry. For this, as discussed above, we need a laplacian. From the above discussion is is clear there are now two available candidates, the quadratic Casimir operator of SO(6) or that of SO(5), which are respectively:

$$\mathrm{C}_2^{\mathrm{SO}(6)} = \frac{1}{2}(AdJ_{AB})^2 \tag{4.6}$$

$$\mathrm{C}_2^{\mathrm{SO}(5)} = \frac{1}{2}(AdJ_{ab})^2 \tag{4.7}$$

If we chose the SO(6) Casimir (which is equally the SU(4) Casimir) we are imposing the geometry of a round $\mathbb{C}\mathrm{P}^3$. But of course for a round S^4 SO(5) symmetry is all we need.

If we take the Spin(5) point of view then an arbitrary matrix can be considered as an element of the vector space $(\frac{L}{2},\frac{L}{2}) \otimes \overline{(\frac{L}{2},\frac{L}{2})}$ which reduces under Spin(5) as:

$$\left(\frac{L}{2},\frac{L}{2}\right) \otimes \left(\frac{L}{2},\frac{L}{2}\right) = \bigoplus_{n=0}^{L}\bigoplus_{m=0}^{n}(n,m)\,. \tag{4.8}$$

with the dimension of (n,m) being [20]

$$\dim(n,m) = \frac{1}{6}\,(2n+3)\,(2m+1)\,((n+1)(n+2) - m(m+1))\,. \tag{4.9}$$

Only the representations $(n,0)$ correspond to functions on S^4, all others are non-S^4 representations.

Equivalently taking the Spin(6) point of view one has the Spin(6) reduction

$$\left(\frac{L}{2},\frac{L}{2},\frac{L}{2}\right) \otimes \left(\frac{L}{2},\frac{L}{2},-\frac{L}{2}\right) = \bigoplus_{n=0}^{L}(n,n,0) \tag{4.10}$$

where

$$\dim(n,n,0) = \frac{1}{6}\,(2n+3)\,(n+1)^2\,(n+2)^2\,. \tag{4.11}$$

Furthermore, one can see that the SO(6) representation $(n,n,0)$ breaks up as a sum of SO(5) representations and we have:

$$(n,n,0) = \bigoplus_{m=0}^{n}(n,m). \tag{4.12}$$

One can gain more insight into the role of these representations and the above decompositions by thinking of the above arguments in terms of polarization tensors.

In our context, an arbitrary matrix, which plays the role of a scalar field, can be decomposed into a sum of orthonormal polarization tensors where the set of polarization tensors carry the representation content (4.8). Thus an arbitrary matrix $M \in \mathrm{Mat}_{d_L}$, can be decomposed as

$$M = \sum_{n=0}^{L} \sum_{m \leq n} M^{(n,m)}_{a_1,a_2,\cdots,a_{n+m}} \mathcal{V}^{(n,m)}_{a_1,a_2,\cdots,a_{n+m}} \tag{4.13}$$

where

$$\mathcal{V}^{(n,m)}_{a_1,a_2\cdots,a_{m+n}} \in \qquad \text{the SO(5) IRR } (n,m) \quad m \leq n \tag{4.14}$$

are the polarization tensors. These are in fact appropriately symmetrized n-th order polynomial of J_a and J_{ab}, with traces removed and with the order of J_{ab} being m. Those that correspond to functions on S^4 are therefore the $\mathcal{V}^{(n,0)}$ and are matrix versions of the S^4 spherical harmonics and the direct analogues of the $\hat{Y}_{lm}$ of [6] but in an orthogonal basis.

In fact, the decomposition (4.13) corresponds to the SU(4) decomposition:

$$\mathbf{d_L} \otimes \overline{\mathbf{d_L}} = \mathbf{1} \oplus \mathbf{15} \oplus \mathbf{84} \oplus \cdots \oplus \mathbf{D_{n-1}} \oplus \mathbf{D_n}$$

where $\mathrm{D_n} = dim(n,0,n) = \frac{1}{12}(2n+3)(n+1)^2(n+2)^2$ with $\mathbf{D_n}$ further decomposed under Spin(5) as

$$\mathbf{D_n} = \bigoplus_{m=0}^{n} (n,m). \tag{4.15}$$

For example the $\mathbf{15}$ breaks up as $\mathbf{15} = \mathbf{5} \oplus \mathbf{10}$ which corresponds to the decomposition of J_{AB} into J_a and J_{ab}. Similarly the $\mathbf{84}$ decomposes as $\mathbf{84} = \mathbf{14} \oplus \mathbf{35} \oplus \mathbf{35'}$ and so on.

Let us now discuss the eigenvalues of the Casimirs on the above representations. We have

$$\mathrm{C}_2^{\mathrm{SO(6)}} \mathcal{V}^{(n,m)} = 2n(n+3)\mathcal{V}^{(n,m)} \tag{4.16}$$

$$\mathrm{C}_2^{\mathrm{SO(5)}} \mathcal{V}^{(n,m)} = \{n(n+3) + m(m+1)\}\, \mathcal{V}^{(n,m)}. \tag{4.17}$$

We see that the operator

$$\mathrm{C}_I = 2\mathrm{C}_2^{\mathrm{SO(5)}} - \mathrm{C}_2^{\mathrm{SO(6)}} \tag{4.18}$$

has eigenvalues

$$\mathrm{C}_I \mathcal{V}^{(n,m)} = 2m(m+1)\mathcal{V}^{(n,m)} \tag{4.19}$$

and is precisely the operator we require to separate the wanted S^4 modes from the unwanted modes. We are therefore in a position to fix the geometry for our fuzzy space in a fashion which will suppress the non-S^4 modes. The desired laplacian will be

$$\Delta_h = \frac{(\mathrm{C}_2^{\mathrm{SO(6)}} + h\mathrm{C}_I)}{2R^2}\,. \tag{4.20}$$

The eigenmatrices of this operator are then the polarization tensors $\mathcal{V}^{(n,m)}$ and we have the eigenvalue equations

$$\Delta_h \mathcal{V}^{(n,m)} = \left\{ \frac{n(n+3) + hm(m+1)}{R^2} \right\} \mathcal{V}^{(n,m)} . \tag{4.21}$$

From the spectrum we see that Δ_h has positive spectrum for $h \in (-1, \infty)$ for all values of L. In fact for L finite the permitted values of h are slightly larger and one can choose $h \in [-(L+2)/(L+1), \infty)$.

Furthermore now that we have identified a laplacian type operator which distinguishes between the S^4 and non-S^4 modes we can further identify the projector that removes the unwanted modes from an arbitrary matrix. It is simply

$$\mathcal{P}_{S^4} = \prod_{n=1}^{L} \prod_{m=1}^{n} \frac{\mathrm{C}_2^{\mathrm{SO}(5)} - \lambda_{n,m}}{\lambda_{n,0} - \lambda_{n,m}} = \prod_{m=1}^{L} \frac{2m(m+1) - \mathrm{C}_I}{2m(m+1)} \tag{4.22}$$

where $\lambda_{n,m}$ are the eigenvalues of $\mathrm{C}_2^{\mathrm{SO}(5)}$ of (4.17).

As mentioned the choice of $\mathrm{C}_2^{\mathrm{SO}(6)}$ as laplacian fixes the geometry to be that of a round $\mathbb{C}\mathrm{P}^3$. Choosing the linear combination (4.20) of the two Casimirs (4.6) and (4.7) still corresponds to a fuzzy approximation of $\mathbb{C}\mathrm{P}^3$ with the parameter h specifying a continuous deformation of its geometry. The set of eigenmatrices and the function algebra is unchanged, only the laplacian and its eigenvalues are deformed in a continuous fashion. The geometry will no longer correspond to a round $\mathbb{C}\mathrm{P}^3$ but rather to a squashed $\mathbb{C}\mathrm{P}^3$ with SO(5) rather than SO(6) symmetry. We will make the geometry more explicit in a subsequent article [19].

We now see that choosing the modification of the action to be of the form

$$S_I[\Phi] = \frac{\mathrm{Tr}}{d_L} \left(\frac{1}{2} \Phi \Delta_I \Phi \right) \qquad \text{where } \Delta_I = \frac{C_I}{2R^2} \tag{4.23}$$

has precisely the desired properties, i.e. of being zero on matrices corresponding to functions on S^4 and otherwise positive. So when the parameter h is made arbitrarily large the non-S^4 modes are suppressed.

5. Scalar field theory on fuzzy 4-sphere

In this section we will summarize the prescription for working with a scalar field on S^4_F. We begin with the round scalar field theory on $\mathbb{C}\mathrm{P}^3$ given by the action,

$$S_0[\Phi] = \frac{R^4}{d_L} \mathrm{Tr} \left(\frac{1}{4R^2} [J_{AB}, \Phi]^\dagger [J_{AB}, \Phi] + V[\Phi] \right) . \tag{5.1}$$

To this we add, the SO(6) non-invariant but SO(5) invariant term

$$S_I[\Phi] = \frac{R^4}{d_L} \mathrm{Tr} \frac{1}{2R^2} \left([J_{ab}, \Phi]^\dagger [J_{ab} \Phi] - \frac{1}{2} [J_{AB} \Phi]^\dagger [J_{AB} \Phi] \right) \tag{5.2}$$

so that we have the overall action $S[\Phi] = S_0[\Phi] + hS_I[\Phi]$. This prescription is equivalent to taking the laplacian which specifies the geometry to be

$$\begin{aligned}\Delta_h\cdot &= \frac{1}{2R^2}\left(\frac{1}{2}[J_{AB},[J_{AB},\cdot]] + h([J_{ab}[J_{ab},\cdot]] - \frac{1}{2}[J_{AB},[J_{AB},\cdot]])\right)\\ &= \frac{1}{2R^2}\left([J_a,[J_a,\cdot]] + \frac{(1+h)}{2}([J_{ab}[J_{ab},\cdot]] - [J_a,[J_a,\cdot]])\right)\end{aligned} \tag{5.3}$$

or equivalently

$$\Delta_h = \frac{1}{2R^2}\left(\mathrm{C}_2^{\mathrm{SO}(6)} + h(2\mathrm{C}_2^{\mathrm{SO}(5)} - \mathrm{C}_2^{\mathrm{SO}(6)})\right) \tag{5.4}$$

which gives a stable theory for all L if $h \in (-1,\infty)$.

This form (5.4) is an interpolation between SO(5) and SO(6) Casimirs and the laplacian is proportional to the SO(6) Casimir for $h = 0$ and the SO(5) Casimir for $h = 1$. The values of h of interest to us are those large and positive since in the quantization of the theory following euclidean functional integral methods, the states unrelated to S^4 then become highly improbable; this is a direct consequence of (4.21) and the expression (3.5) for the probability $\mathcal{P}[\Phi]$.

Note: we have not specified the potential of the model since the above prescription is independent of the potential. The most obvious model to consider would be a quartic potential, since this is relevant to the Higgs sector of the standard model.

6. Conclusions

We have presented a solution to the elimination of non-S^4 modes in the fuzzy approach to S^4. The solution was to modify the laplacian of the scalar action. The modification was to choose the overall laplacian to be a perturbation of the round one on $\mathbb{C}\mathrm{P}^3$ retaining only the SO(5) symmetry of S^4 and proportional to $\mathrm{C}_I = 2\mathrm{C}_2^{\mathrm{SO}(5)} - \mathrm{C}_2^{\mathrm{SO}(6)}$. The resulting laplacian operator, which specifies the geometry in the fuzzy models, takes the form (5.3) or (5.4) and has spectrum

$$\lambda_{n,m} = \frac{n(n+3) + hm(m+1)}{R^2} \qquad n = 0, 1, \ldots, L \quad \text{and} \quad m \leq n\,. \tag{6.1}$$

It is a positive operator for all L if $h \in (-1,\infty)$.

From our study of the representation content in section 4 it was possible to construct the projector $\mathcal{P}_{S^4}$, eq. (4.22), which eliminates the non-S^4 modes. This is precisely the projector necessary for the non-associative algebra discussed in [11]. We have not pursued this algebra due to its complications.

As we will see in a subsequent article [19], using coherent state techniques it is possible to extract the metric (and hence the geometry) of the above spaces from the laplacian. We will in fact show that the space in all cases is $\mathbb{C}\mathrm{P}^3$ and that this space can be viewed as either the orbit $\mathrm{SU}(4)/\mathrm{U}(3)$ or the orbit $\mathrm{Spin}(5)/\mathrm{SU}(2)\times\mathrm{U}(1)$. It therefore in general permits an SO(5) invariant metric when viewed as a Spin(5) orbit, but this can be expanded to a maximal SO(6) invariance, which we refer to as the round $\mathbb{C}\mathrm{P}^3$. When we have only SO(5) symmetry we have what we call a squashed $\mathbb{C}\mathrm{P}^3$. In fact it is possible to show that $\mathbb{C}\mathrm{P}^3$ is locally the direct product $S^4\times S^2$ and globally an S^2 bundle over S^4 and that the parameter h is a measure of the size of the S^2 fibres. In fact we will show [19]

$$\frac{R^2_{S^2}}{R^2}=\frac{1}{1+h} \tag{6.2}$$

so the value $h\to\infty$ corresponds to shrinking the S^2 fibres to zero size, while $h\to -1$ corresponds to making the fibres infinitely large.

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